Imperial/TP/2013/mjd/01

A magic square from Yang-Mills squared

L. Borsten,* M. J. Duff,† L. J. Hughes,‡ and S. Nagy§
Theoretical Physics, Blackett Laboratory, Imperial College London, London SW7 2AZ, United Kingdom

(Dated: February 14, 2013)

We give a division algebra $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ description of $D = 3$ Yang-Mills with $\mathcal{N} = 1, 2, 4, 8$ and hence, by tensoring left and right multiplets, a magic square $\mathbb{RR}$, $\mathbb{CR}$, $\mathbb{CC}$, $\mathbb{HR}$, $\mathbb{HC}$, $\mathbb{HH}$, $\mathbb{OR}$, $\mathbb{OC}$, $\mathbb{OH}$, $\mathbb{OO}$ description of $D = 3$ supergravity with $\mathcal{N} = 2, 3, 4, 5, 6, 8, 9, 10, 12, 16$.

PACS numbers: 04.65.+e, 02.20.Sv, 11.30.-j
Keywords: U-duality, magic square, Yang-Mills squared

INTRODUCTION

The octonions occupy a privileged position as the largest of the division algebras $\mathbb{A}$: reals $\mathbb{R}$, complexes $\mathbb{C}$, quaternions $\mathbb{H}$ and octonions $\mathbb{O}$. They provide an intuitive basis for the exceptional Lie groups. For example, the smallest exceptional group G_2 can be understood as the set of automorphims preserving the octonionic product. Efforts to understand the remaining exceptional groups geometrically in terms of octonions resulted in the Freudenthal-Rozenfeld-Tits magic square [1–6] presented in Table I[1]. Despite much effort, however, it is fair to say that the ultimate physical significance of octonions and the magic square remains an enigma.

$\mathbb{A}_L/\mathbb{A}_R$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	SL(2, $\mathbb{R}$)	SU(2, 1)	USp(4, 2)	$F_{4(-20)}$
$\mathbb{C}$	SU(2, 1)	SU(2, 1) × SU(2, 1)	SU(4, 2)	$E_{6(-14)}$
$\mathbb{H}$	USp(4, 2)	SU(4, 2)	SO(8, 4)	$E_{7(-5)}$
$\mathbb{O}$	$F_{4(-20)}$	$E_{6(-14)}$	$E_{7(-5)}$	$E_{8(8)}$

TABLE I. Magic square

In apparently different developments, a recurring theme in attempts to understand the quantum theory of gravity is the idea of "Gravity as the square of Yang-Mills". This idea of tensoring left (L) and right (R) multiplets appears in many different (but sometimes overlapping) guises: KLT relations in string theory [10], $D = 10$ dimensional Type IIA and IIB supergravity (SG) multiplets from $D = 10$ super Yang-Mills (SYM) multiplets [15], asymmetric orbifold contructions [16], gravity anomalies from gauge anomalies [17], (super)gravity scattering amplitudes from those of (super) Yang-Mills [11–14] in various dimensions etc.

In the supersymmetric context it is not difficult to see that the amount of supersymmetry is given by

$$[\mathcal{N}_L \text{ SYM}] \otimes [\mathcal{N}_R \text{ SYM}] \rightarrow [\mathcal{N} = \mathcal{N}_L + \mathcal{N}_R \text{ SG}], \quad (1)$$

but it is harder to see how the other gravitational symmetries arise from those of Yang-Mills. In particular, supergravities are characterized by non-compact global symmetries G (the so-called U-dualities) with local compact subgroups H, for example $G = E_{7(7)}$ and $H = \mathrm{SU}(8)$ for $\mathcal{N} = 8$ supergravity in $D = 4$; whereas the Yang-Mills we start with has global R-symmetries, for example $\mathrm{R} = \mathrm{SU}(4)$ for $\mathcal{N} = 4$ in $D = 4$. See [18] for an approach linking SU(4) to SU(8) based on scattering amplitudes.

In the present paper we focus initially on $D = 3$. This is not only intrinsically interesting [14, 19–21], but also throws light on higher-dimensional theories to which it is related by dimensional reduction. First we give a division algebra $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ description of $D = 3$ Yang-Mills with $\mathcal{N} = 1, 2, 4, 8$, which is of interest in its own right. More remarkable, however, is that tensoring left and right multiplets yields a magic square $\mathbb{RR}$, $\mathbb{CR}$, $\mathbb{CC}$, $\mathbb{HR}$, $\mathbb{HC}$, $\mathbb{HH}$, $\mathbb{OR}$, $\mathbb{OC}$, $\mathbb{OH}$, $\mathbb{OO}$ description of $D = 3$ supergravity with $\mathcal{N} = 2, 3, 4, 5, 6, 8, 9, 10, 12, 16$, as presented in Table II. For $\mathcal{N} > 8$ the multiplets are those of pure supergravity; for $\mathcal{N} \leq 8$ supergravity is coupled to matter. In both cases the field content is such that the U-dualities exactly match the groups of Table I.

Thus not only do $D = 3$ supergravities fill out a magic square but their field content and, hence, symmetries are *derived* from squaring Yang-Mills.

MAGIC SQUARE

Magic squares are based on the four division algebras, $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$, which are of dimension 1, 2, 4 and 8, respectively[2]. They can be built, one-by-one, using the Cayley-Dickson doubling procedure starting with $\mathbb{R}$. The reals are ordered, commutative and associative. With

[1] There are a variety of magic squares in which different real forms appear. See [7] for a comprehensive account in the context of supergravity. Famously, the $\mathbb{C}, \mathbb{H}$, and $\mathbb{O}$ rows of one example describe the U-dualities of the aptly named "magic" supergravities in $D = 5, 4, 3$ respectively [8, 9]. In this paper we instead demonstrate the novel appearance of the magic square of Table I in conventional $D = 3$ supergravities.

[2] One can also use their split (non-division) cousins to obtain different real forms. See [7, 22] and the reference therein for details.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathcal{N}=2, f=4$ $G=\mathrm{SL}(2,\mathbb{R})$, dim 3 $H=\mathrm{SO}(2)$, dim 1	$\mathcal{N}=3, f=8$ $G=\mathrm{SU}(2,1)$, dim 8 $H=\mathrm{SU}(2)\times\mathrm{SO}(2)$, dim 4	$\mathcal{N}=5, f=16$ $G=\mathrm{USp}(4,2)$, dim 21 $H=\mathrm{USp}(4)\times\mathrm{USp}(2)$, dim 13	$\mathcal{N}=9, f=32$ $G=F_{4(-20)}$, dim 52 $H=\mathrm{SO}(9)$, dim 36
$\mathbb{C}$	$\mathcal{N}=3, f=8$ $G=\mathrm{SU}(2,1)$, dim 8 $H=\mathrm{SU}(2)\times\mathrm{SO}(2)$, dim 4	$\mathcal{N}=4, f=16$ $G=\mathrm{SU}(2,1)^2$, dim 16 $H=\mathrm{SU}(2)^2\times\mathrm{SO}(2)^2$, dim 8	$\mathcal{N}=6, f=32$ $G=\mathrm{SU}(4,2)$, dim 35 $H=\mathrm{SU}(4)\times\mathrm{SU}(2)\times\mathrm{SO}(2)$, dim 19	$\mathcal{N}=10, f=64$ $G=E_{6(-14)}$, dim 78 $H=\mathrm{SO}(10)\times\mathrm{SO}(2)$, dim 46
$\mathbb{H}$	$\mathcal{N}=5, f=16$ $G=\mathrm{USp}(4,2)$, dim 21 $H=\mathrm{USp}(4)\times\mathrm{USp}(2)$, dim 13	$\mathcal{N}=6, f=32$ $G=\mathrm{SU}(4,2)$, dim 35 $H=\mathrm{SU}(4)\times\mathrm{SU}(2)\times\mathrm{SO}(2)$, dim 19	$\mathcal{N}=8, f=64$ $G=\mathrm{SO}(8,4)$, dim 66 $H=\mathrm{SO}(8)\times\mathrm{SO}(4)$, dim 34	$\mathcal{N}=12, f=128$ $G=E_{7(-5)}$, dim 133 $H=\mathrm{SO}(12)\times\mathrm{SO}(3)$, dim 69
$\mathbb{O}$	$\mathcal{N}=9, f=32$ $G=F_{4(-20)}$, dim 52 $H=\mathrm{SO}(9)$, dim 36	$\mathcal{N}=10, f=64$ $G=E_{6(-14)}$, dim 78 $H=\mathrm{SO}(10)\times\mathrm{SO}(2)$, dim 46	$\mathcal{N}=12, f=128$ $G=E_{7(-5)}$, dim 133 $H=\mathrm{SO}(12)\times\mathrm{SO}(3)$, dim 69	$\mathcal{N}=16, f=256$ $G=E_{8(8)}$, dim 248 $H=\mathrm{SO}(16)$, dim 120

TABLE II. Magic square of $D=3$ supergravity theories. The first row of each entry indicates the amount of supersymmetry $\mathcal{N}$ and the total number of degrees of freedom f. The second (third) row indicates the U-duality group G (the maximal compact subgroup $H \subset G$) and its dimension. The scalar fields in each case parametrise the coset G/H, where $\dim_{\mathbb{R}}(G/H) = f/2$.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	SO(2)	SO(3) × SO(2)	SO(5) × SO(3)	SO(9)
$\mathbb{C}$	SO(3) × SO(2)	$[\mathrm{SO}(3)\times\mathrm{SO}(2)]^2$	SO(6) × SO(3)	SO(10) × SO(2)
$\mathbb{H}$	SO(5) × SO(3)	SO(6) × SO(3)	SO(8) × SO(4)	SO(12) × SO(3)
$\mathbb{O}$	SO(9)	SO(10) × SO(2)	SO(12) × SO(3)	SO(16)

TABLE III. Magic square of maximal compact subgroups.

each doubling one such property is lost: $\mathbb{C}$ is commutative and associative, $\mathbb{H}$ is associative, $\mathbb{O}$ is *non-associative*.

An element $x \in \mathbb{O}$ may be written $x = x^a e_a$, where $a = 0, \ldots, 7$, $x^a \in \mathbb{R}$ and $\{e_a\}$ is a basis with one real $e_0 = 1$ and seven $e_i, i = 1, \ldots, 7$, imaginary elements. The octonionic conjugation is denoted by e_a^*, where $e_0^* = e_0$ and $e_i^* = -e_i$. The octonionic multiplication rule is,

$$e_a e_b = (\delta_{a0}\delta_{bc} + \delta_{0b}\delta_{ac} - \delta_{ab}\delta_{0c} + C_{abc})\, e_c, \qquad (2)$$

where C_{abc} is totally antisymmetric such that $C_{0bc} = 0$. The non-zero C_{ijk} are given by the Fano plane, see [23].

A natural inner product on $\mathbb{A}$ is defined by

$$\langle x|y\rangle := \frac{1}{2}(x\overline{y} + y\overline{x}) = x^a y^b \delta_{ab}. \qquad (3)$$

To understand the symmetries of the magic square and its relation to SYM we shall need in particular two algebras defined on $\mathbb{A}$. First, the norm-preserving algebra,

$$\mathfrak{so}(\mathbb{A}) := \{D \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{A}) | \langle Dx|y\rangle + \langle x|Dy\rangle = 0\}, \qquad (4)$$

isomorphic to $\mathfrak{so}(\dim_{\mathbb{R}} \mathbb{A})$. Second, the *triality* algebra

$$\mathfrak{tri}(\mathbb{A}) := \{(A, B, C) | A(xy) = (Bx)y + x(Cy)\} \qquad (5)$$

where $A, B, C \in \mathfrak{so}(\mathbb{A})$. For $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ we have $\mathfrak{tri}(\mathbb{A}) \cong \emptyset, \mathfrak{so}(2) \oplus \mathfrak{so}(2), \mathfrak{so}(3) \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(3), \mathfrak{so}(8)$ [22].

The specific magic square presented in Table I was first obtained in [7] using a version of the Tits construction based on a *Lorentzian* Jordan algebra. The two algebras $\mathbb{A}_L, \mathbb{A}_R$ enter this definition on distinct footings; the "magic" of the square is its symmetry under the exchange $\mathbb{A}_L \leftrightarrow \mathbb{A}_R$, which is obscured by their undemocratic treatment.

For the purposes of squaring SYM a manifestly $\mathbb{A}_L \leftrightarrow \mathbb{A}_R$ symmetric formulation of the square is required. This is achieved by adapting the *triality algebra* construction introduced by Barton and Sudbery [22]. Our definition[3] of Table I is given by,

$$\mathfrak{L}_3(\mathbb{A}_L, \mathbb{A}_R) \cong \mathfrak{tri}(\mathbb{A}_L) \oplus \mathfrak{tri}(\mathbb{A}_R) + 3(\mathbb{A}_L \otimes \mathbb{A}_R). \qquad (6)$$

We shall also need a magic square of the maximal compact subalgebras of Table I, given in Table III. This is given by the *reduced* triality construction,

$$\mathfrak{L}_1(\mathbb{A}_L, \mathbb{A}_R) := \mathfrak{tri}(\mathbb{A}_L) \oplus \mathfrak{tri}(\mathbb{A}_R) + (\mathbb{A}_L \otimes \mathbb{A}_R), \qquad (7)$$

which is easily obtained from (6).

$\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ DESCRIPTION OF $D = 3, \mathcal{N} = 1, 2, 4, 8$ YANG-MILLS

The $D = 3$, $\mathcal{N} = 8$ SYM Lagrangian is given by

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F^A_{\mu\nu}F^{A\mu\nu} - \frac{1}{2}D_\mu\phi^A_i D^\mu\phi^A_i + i\bar{\lambda}^A_a\gamma^\mu D_\mu\lambda^A_a \\ &-\frac{1}{4}g^2 f_{BC}{}^A f_{DE}{}^A \phi^B_i\phi^D_i\phi^C_j\phi^E_j - g f_{BC}{}^A \phi^B_i \bar{\lambda}^{Aa}\Gamma^i_{ab}\lambda^{Cb},\end{aligned}$$

where Γ^i_{ab}, $i = 1, \ldots, 7$, $a, b = 0, \ldots, 7$, belongs to the SO(7) Clifford algebra. The key observation is that

[3] Note, this is not quite the triality construction as defined in [22]. We will not present the details here, but it can be easily obtained by making a slight modification to the commutators in $3(\mathbb{A}_L \otimes \mathbb{A}_R)$ w.r.t. those appearing in [22]. Here, we have used $\oplus$ and $+$ to distinguish the direct sum between Lie algebras and vector spaces, i.e. only if $[\mathfrak{g}, \mathfrak{h}] = 0$ do we use $\mathfrak{g} \oplus \mathfrak{h}$.

$\mathbb{A}_L/\mathbb{A}_R$	$A_\mu(R) \in \mathrm{Re}\mathbb{A}_R$	$\phi(R) \in \mathrm{Im}\mathbb{A}_R$	$\lambda(R) \in \mathbb{A}_R$
$A_\mu(L) \in \mathrm{Re}\mathbb{A}_L$	$g_{\mu\nu} + \varphi \in \mathrm{Re}\mathbb{A}_L \otimes \mathrm{Re}\mathbb{A}_R$	$\varphi \in \mathrm{Re}\mathbb{A}_L \otimes \mathrm{Im}\mathbb{A}_R$	$\Psi_\mu + \chi \in \mathrm{Re}\mathbb{A}_L \otimes \mathbb{A}_R$
$\phi(L) \in \mathrm{Im}\mathbb{A}_L$	$\varphi \in \mathrm{Im}\mathbb{A}_L \otimes \mathrm{Re}\mathbb{A}_R$	$\varphi \in \mathrm{Im}\mathbb{A}_L \otimes \mathrm{Im}\mathbb{A}_R$	$\chi \in \mathrm{Im}\mathbb{A}_L \otimes \mathbb{A}_R$
$\lambda(L) \in \mathbb{A}_L$	$\Psi_\mu + \chi \in \mathbb{A}_L \otimes \mathrm{Re}\mathbb{A}_R$	$\chi \in \mathbb{A}_L \otimes \mathrm{Im}\mathbb{A}_R$	$\varphi \in \mathbb{A}_L \otimes \mathbb{A}_R$

TABLE IV. Tensor product of left/right ($\mathbb{A}_L/\mathbb{A}_R$) SYM multiplets, using $\mathrm{SO}(1,2)$ spacetime reps and dualising all p-forms.

this gamma matrix can be represented by the octonionic structure constants,

$$\Gamma^i_{ab} = i(\delta_{bi}\delta_{a0} - \delta_{b0}\delta_{ai} + C_{iab}), \tag{8}$$

which allows us to rewrite the action over octonionic fields. If we replace $\mathbb{O}$ with a general division algebra $\mathbb{A}$, the result is $\mathcal{N} = 1, 2, 4, 8$ over $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F^A_{\mu\nu}F^{A\mu\nu} - \frac{1}{2}D_\mu\phi^{*A}D^\mu\phi^A + i\bar{\lambda}^A\gamma^\mu D_\mu\lambda^A \\ &-\frac{1}{4}g^2 f_{BC}{}^A f_{DE}{}^A \langle\phi^B|\phi^D\rangle\langle\phi^C|\phi^E\rangle \\ &+\frac{i}{2}g f_{BC}{}^A\left((\bar{\lambda}^A\phi^B)\lambda^C - \bar{\lambda}^A(\phi^{*B}\lambda^C)\right),\end{aligned}$$

where $\phi = \phi^i e_i$ is an $\mathrm{Im}\mathbb{A}$-valued scalar field, $\lambda = \lambda^a e_a$ is an $\mathbb{A}$-valued two-component spinor and $\bar{\lambda} = \bar{\lambda}^a e^*_a$. Note, since λ^a is anti-commuting we are dealing with the algebra of octonions defined over the Grassmanns.

The supersymmetry transformations in this language are given by

$$\begin{aligned}\delta\lambda^A &= \frac{1}{2}(F^{A\mu\nu} + \varepsilon^{\mu\nu\rho}D_\rho\phi^A)\sigma_{\mu\nu}\epsilon + \frac{1}{2}g f_{BC}{}^A\phi^B_i\phi^C_j\sigma_{ij}\epsilon, \\ \delta A^A_\mu &= \frac{i}{2}(\bar{\epsilon}\gamma_\mu\lambda^A - \bar{\lambda}^A\gamma_\mu\epsilon), \\ \delta\phi^A &= \frac{i}{2}e_i[(\bar{\epsilon}e_i)\lambda^A - \bar{\lambda}^A(e_i\epsilon)],\end{aligned} \tag{9}$$

where ϵ is an $\mathbb{A}$-valued two-component spinor and $\sigma_{\mu\nu}$ are the generators of $\mathrm{SL}(2,\mathbb{R}) \cong \mathrm{SO}(1,2)$. The σ_{ij} generate $\mathrm{SO}(\mathrm{Im}\mathbb{A})$ and are proportional to the identity as 2×2 matrices, but act as operators on $\mathbb{A}$ itself. In the octonionic case these operators are best expressed by decomposing $\mathfrak{so}(\mathrm{Im}\mathbb{O})$ into its $\mathfrak{g}_2$ subalgebra to give $\sigma_{ij} = \Gamma_{ij} + \Sigma_{ij}$, with

$$\begin{aligned}\Gamma_{ij} &= \left(\frac{1}{2}[e_i, e_j, \ \cdot\ ] - \frac{1}{6}\,[[e_i, e_j], \ \cdot\ ]\right)\mathbb{1}, \\ \Sigma_{ij} &= -\frac{1}{12}\,[[e_i, e_j], \ \cdot\ ]\,\mathbb{1},\end{aligned} \tag{10}$$

where $[\cdot,\cdot,\cdot]$ is the associator: $[a,b,c] := (ab)c - a(bc)$. Under $\mathrm{SO}(7) \supset G_2$, $\mathbf{21} \to \mathbf{14} + \mathbf{7}$, the Γ_{ij} and Σ_{ij} and correspond to the $\mathbf{14}$ and $\mathbf{7}$, respectively. For $\mathbb{A} = \mathbb{H}$ the associator vanishes and the σ_{ij} generate $\mathrm{SO}(3)$, the automorphism group of the quaternions; for $\mathbb{A} = \mathbb{R}, \mathbb{C}$ the σ_{ij} trivially vanish.

SQUARING YANG-MILLS

Having cast the magic square in terms of a manifestly $\mathbb{A}_L \leftrightarrow \mathbb{A}_R$ symmetric triality algebra construction, and having written $\mathcal{N} = 1, 2, 4, 8$ SYM in terms of fields valued in $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ we shall now obtain the magic square of supergravities in Table II, with symmetries G (Table I) and H (Table III), by "squaring" $\mathcal{N} = 1, 2, 4, 8$ SYM.

Taking a left SYM multiplet $\{A_\mu(L) \in \mathrm{Re}\mathbb{A}_L, \phi(L) \in \mathrm{Im}\mathbb{A}_L, \lambda(L) \in \mathbb{A}_L\}$ and tensoring it with a right multiplet $\{A_\mu(R) \in \mathrm{Re}\mathbb{A}_R, \phi(R) \in \mathrm{Im}\mathbb{A}_R, \lambda(R) \in \mathbb{A}_R\}$ we obtain the field content of a supergravity theory valued in both $\mathbb{A}_L$ and $\mathbb{A}_R$. See Table IV. Note, the left/right SYM R-symmetries act on each slot of the $\mathbb{A}_L, \mathbb{A}_R$ tensor products.

Grouping spacetime fields of the same type we find,

$$g_{\mu\nu} \in \mathbb{R}, \quad \Psi_\mu \in \begin{pmatrix}\mathbb{A}_L \\ \mathbb{A}_R\end{pmatrix}, \quad \varphi, \chi \in \begin{pmatrix}\mathbb{A}_L \otimes \mathbb{A}_R \\ \mathbb{A}_L \otimes \mathbb{A}_R\end{pmatrix}. \tag{11}$$

The $\mathbb{R}$-valued graviton and $\mathbb{A}_L \oplus \mathbb{A}_R$-valued gravitino carry no degrees of freedom. The $(\mathbb{A}_L \otimes \mathbb{A}_R)^2$-valued scalar and Majorana spinor each have $2(\dim\mathbb{A}_L \times \dim\mathbb{A}_R)$ degrees of freedom.

As we have already mentioned, the $\mathcal{N} > 8$ supergravities in $D = 3$ are unique, all fields belonging to the gravity multiplet, while those with $\mathcal{N} \leq 8$ may be coupled to k additional matter multiplets [20, 21]. The real miracle is that tensoring left and right SYM multiplets yields the field content of $\mathcal{N} = 2, 3, 4, 5, 6, 8$ supergravity with $k = 1, 1, 2, 1, 2, 4$: just the right matter content to produce the U-duality groups appearing in Table I.

The largest linearly realised global symmetry of these theories is H, which has Lie algebra given by the reduced triality construction (7). Consequently, we expect the fields in (11) to carry linear representations of H. The metric is a singlet, while Ψ_μ, φ and χ transform as a vector, spinor and conjugate spinor, respectively. Fortunately, $\mathbb{A}_L \oplus \mathbb{A}_R$ and $(\mathbb{A}_L \otimes \mathbb{A}_R)^2$ are precisely the representation spaces of the vector and (conjugate) spinor. For example, in the maximal case of $\mathbb{A}_L, \mathbb{A}_R = \mathbb{O}$, we have the $\mathbf{16}, \mathbf{128}$ and $\mathbf{128}'$ of $\mathrm{SO}(16)$. The distinction between spinor and conjugate spinor in terms of $(\mathbb{O}_L \otimes \mathbb{O}_R)^2$ is encoded in the division-algebraic realisation of the Lie algebra action, which is inherited from the left/right SYM. For example, consider $x, y \in \mathbb{O}$ transforming respectively as the $\mathbf{8}_s$ and $\mathbf{8}_c$ of $\mathrm{SO}(8)$. A subset of $\mathrm{SO}(8)$ genera-

tors are given by left/right multiplications with elements $a \in \mathrm{Im}\mathbb{O}$ under which $x \mapsto ax$ implies $y \mapsto ya$.

The U-dualities G are realised non-linearly on the scalars, which parametrise the symmetric spaces G/H. This can be understood using the remarkable identity relating the projective planes over $(\mathbb{A}_L \otimes \mathbb{A}_R)^2$ to G/H,

$$(\mathbb{A}_L \otimes \mathbb{A}_R)\mathbb{P}^2 \cong G/H. \tag{12}$$

The scalar fields may be regarded as points in division-algebraic projective planes. The tangent space at any point of $(\mathbb{A}_L \otimes \mathbb{A}_R)\mathbb{P}^2$ is just $(\mathbb{A}_L \otimes \mathbb{A}_R)^2$, the required representation space of H. The *Cayley plane* $\mathbb{O}\mathbb{P}^2$, with isometry group $F_{4(-52)}$, is a classic example: $F_{4(-52)}/\mathrm{Spin}(9) \cong (\mathbb{R}\otimes\mathbb{O})\mathbb{P}^2 = \mathbb{O}\mathbb{P}^2$. The tangent space at any point of $\mathbb{O}\mathbb{P}^2$ is $\mathbb{O}^2$, the spinor of Spin(9) as required. Note, the cases $(\mathbb{C}\otimes\mathbb{O})\mathbb{P}^2, (\mathbb{H}\otimes\mathbb{O})\mathbb{P}^2, (\mathbb{O}\otimes\mathbb{O})\mathbb{P}^2$ are not strictly speaking projective spaces, but may nevertheless be identified with G/H [3, 23, 24].

MAGIC PYRAMID

We have given an $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ description of $\mathcal{N} = 1, 2, 4, 8$ SYM. Tensoring left($\mathbb{A}_L$)/right($\mathbb{A}_R$) multiplets we built an $(\mathbb{A}_L, \mathbb{A}_R)$ magic square of $D = 3$, $\mathcal{N} = 2, 3, 4, 5, 6, 8, 9, 10, 12, 16$ supergravity.

However, it has also been known for some time now that $\mathcal{N} = 1$ SYM in $D = 3, 4, 6, 10$ has a concise division-algebraic formulation [25–28]. Indeed, a simple alternative way to obtain the $\mathcal{N} = 1, 2, 4, 8$ theories over $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ in $D = 3$ is to start from this description of $\mathcal{N} = 1$ SYM in $D = 3, 4, 6, 10$ and dimensionally reduce on a torus. The theories in $D = \dim(\mathbb{A}) + 2$ exploit the $\mathfrak{so}(1, \dim(\mathbb{A}) + 1) \cong \mathfrak{sl}(2, \mathbb{A})$ Lie algebra isomorphisms (in the sense of [29]). Reducing to $D = 3$ decomposes $\mathrm{SL}(2, \mathbb{A}) \supset \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO}(\mathrm{Im}\mathbb{A})$; the algebra that started out life determining D now fixes $\mathcal{N}$.

Hence, we may specify a $(D, \mathcal{N})$ SYM theory using a pair of division algebras $\mathbb{A}^2 = (\mathbb{A}_1, \mathbb{A}_2)$, which are constrained to satisfy $\dim \mathbb{A}_1 + \dim \mathbb{A}_2 \leq 9$. Therefore, tensoring $(\mathbb{A}^2_L)/(\mathbb{A}^2_R)$ SYM multiplets, we can build supergravity theories in $D = 3, 4, 6, 10$ with $\mathcal{N} = 16, \ldots, 2$.

Since the first slots of $(\mathbb{A}^2_L)$ and $(\mathbb{A}^2_R)$ must be correlated and $\dim \mathbb{A}_{1L/R} + \dim \mathbb{A}_{2L/R} \leq 9$, we obtain a *supergravity magic pyramid*, of which the square described here is only the base. The tip is type II supergravity in $D = 10$. The details of this pyramid will be presented elsewhere.

The work of LB is supported by an Imperial College Junior Research Fellowship. The work of MJD is supported by the STFC under rolling grant ST/G000743/1. LJH and SN are supported by STFC PhD studentships.

* leron.borsten@imperial.ac.uk
† m.duff@imperial.ac.uk
‡ leo.hughes07@imperial.ac.uk
§ s.nagy11@imperial.ac.uk

[1] H. Freudenthal, Nederl. Akad. Wetensch. Proc. Ser. **57**, 218 (1954).
[2] H. Freudenthal, Nederl. Akad. Wetensch. Proc. Ser. **A62**, 466 (1959).
[3] H. Freudenthal, Adv. Math. **1**, 145 (1964).
[4] B. A. Rosenfeld, Dokl. Akad. Nauk. SSSR **106**, 600 (1956).
[5] J. Tits, Indag. Math. **28**, 223 (1966).
[6] E. B. Vinberg, Tr. Semin. Vektorn. Tr. Semin. Vektorn. Tensorn. Anal. **13** (1966).
[7] S. L. Cacciatori, B. L. Cerchiai, and A. Marrani, (2012), arXiv:1208.6153.
[8] M. Günaydin, G. Sierra, and P. K. Townsend, Nucl. Phys. **B242**, 244 (1984).
[9] M. Günaydin, G. Sierra, and P. K. Townsend, Phys. Lett. **B133**, 72 (1983).
[10] H. Kawai, D. Lewellen, and S. Tye, Nucl.Phys. **B269**, 1 (1986).
[11] Z. Bern, J. Carrasco, and H. Johansson, Phys.Rev. **D78**, 085011 (2008), arXiv:0805.3993.
[12] Z. Bern, J. J. M. Carrasco, and H. Johansson, Phys.Rev.Lett. **105**, 061602 (2010), arXiv:1004.0476.
[13] Z. Bern, T. Dennen, Y.-t. Huang, and M. Kiermaier, Phys.Rev. **D82**, 065003 (2010), arXiv:1004.0693.
[14] Y.-t. Huang and H. Johansson, (2012), arXiv:1210.2255.
[15] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory vol. 1: Introduction*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, UK, 1987) 469 p.
[16] A. Sen and C. Vafa, Nucl. Phys. **B455**, 165 (1995), arXiv:hep-th/9508064.
[17] I. Antoniadis, E. Gava, and K. Narain, Phys.Lett. **B283**, 209 (1992), arXiv:hep-th/9203071.
[18] M. Bianchi, H. Elvang, and D. Z. Freedman, JHEP **0809**, 063 (2008), arXiv:0805.0757.
[19] S. Deser, R. Jackiw, and S. Templeton, Annals Phys. **140**, 372 (1982).
[20] N. Marcus and J. H. Schwarz, Nucl.Phys. **B228**, 145 (1983).
[21] B. de Wit, A. Tollsten, and H. Nicolai, Nucl.Phys. **B392**, 3, hep-th/9208074.
[22] C. H. Barton and A. Sudbery, Adv. in Math. **180**, 596 (2003), arXiv:math/0203010.
[23] J. C. Baez, Bull. Amer. Math. Soc. **39**, 145 (2001), arXiv:math/0105155.
[24] J. Landsberg and L. Manivel, Journal of Algebra **239**, 477 (2001).
[25] T. Kugo and P. K. Townsend, Nucl. Phys. **B221**, 357 (1983).
[26] M. Duff, Class.Quant.Grav. **5**, 189 (1988).
[27] J. M. Evans, Nucl. Phys. **B298**, 92 (1988).
[28] J. C. Baez and J. Huerta, in *Superstrings, Geometry, Topology, and C*-Algebras, eds. R. Doran, G. Friedman and J. Rosenberg, Proc. Symp. Pure Math*, Vol. 81 (2009) pp. 65–80, arXiv:0909.0551.
[29] A. Sudbery, J. Phys. **A17**, 939 (1984).